OPTIMALITY PROPERTIES OF A SPECIAL ASSIGNMENT PROBLEM

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Research Report

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ABSTRACT

In this paper, it is shown that if the cost matrix of an assignment problem has the following property $c_{ij} = |j - i|$ then any basic feasible solution is optimal if and only if its unit components belong to two well defined symmetric regions. The matrix with above mentioned property is called the "REORDERING MATRIX," because it arose for the first time in the reordering of nodes of a critical path and other acyclic network problems.

OPTIMALITY PROPERTIES OF A SPECIAL ASSIGNMENT PROBLEM

I. Introduction

Assume we want to order the nodes of a network in such a way that for every arc (i, j), which belongs to the topology of the network, $i < j$.* In case of a large network, this may not be possible to do manually, as it generates a large permutation problem. An algorithm to reorder the nodes of the network is given in [1], where the number of steps involved is related to the magnitude of divergence of the node order of a network, where divergence is defined to be

$$S = \sum_{i=1}^{m} |i - a_i|$$

where a_i is the initial order of the node having final order i.

In order to fix an upper bound to the divergence of the network, we have to determine the maximum value of S over all possible m-tuples a.

We will now show that this problem is equivalent to the classical assignment problem.

*This problem is important because it reduces significantly the work involved in logical search for a critical path problem. It is impossible to perform this node ordering when loops are present in the network.

II. Mathematical Formulation

Let $\mathcal{A} = \left\{ a \,\middle|\, \begin{array}{l} a \text{ is an m-tuple}^{*} \text{ chosen without repetition from} \\ \text{the set } \{1, 2, \ldots, m\} \text{ e.g., } a = (2, m, 10, \ldots, 1, 5) \end{array} \right\}$.

Note that $\mathcal{A}$ has $m!$ nonidentical components.

Then, the problem becomes

(1) $$\max_{a \in \mathcal{A}} S = \sum_{i=1}^{m} |i - a_i| \quad ,$$

where the m-tuples "a" are chosen without repetition from the set of m first integers, to each integer k corresponding one and only one a_i for each m-tuple "a" . It is now easy to see that all the components of each possible summation $\sum_{i=1}^{m} |i - a_i|$ are of the form $|i - j|$ where $i = 1, \ldots, m$; $j = 1, \ldots, m$. We can list all these possible values under matrix form where each entry is equal to the absolute difference between its row number and column number.

Define $c_{ij} = |i - j|$ and $C = \{c_{ij}\}$. Any possible S can be found by summing up m entries of the matrix selected by picking one and only one element from each row and each column. For example

i	$a_i = j$			
1	c_{11}	c_{12}	c_{13}	c_{14}
2	c_{21}	c_{22}	c_{23}	c_{24}
3	c_{31}	c_{32}	c_{33}	c_{34}
4	c_{41}	c_{42}	c_{43}	c_{44}

* m-tuple "a" has m elements, where a_i $(i = 1, \ldots, m)$ is the i^{th} element.

A possible $S = c_{12} + c_{23} + c_{31} + c_{44}$ corresponding to the 4-tuple "a" = (2, 3, 1, 4) .

Determining the max S is equivalent to finding some feasible combination of the entries of the C matrix (i. e., one and only one entry in each row and each column) such that their sum is maximum. Thus, the problem reduces to

$$\text{maximize } S = \sum_{ij} c_{ij} x_{ij}$$

Subject to:

$$\sum_{j} x_{ij} = 1 \qquad i = 1, \ldots, m$$

$$\sum_{i} x_{ij} = 1 \qquad j = 1, \ldots, m \tag{2}$$

$$x_{ij} \geq 0 \qquad x_{ij} \text{ are integers} \quad ,$$

which is the classical assignment problem formulation.

Our aim is to prove,in this particular set-up, that this optimum for S can be achieved by selecting any feasible solution such that all its components belong to two symmetric regions of the matrix and that no optimal solution can be found if one or more components do not belong to these two regions.

III. The Optimal Region

In the reordering matrix let us define the two following sets of entries.

$$O_1^* = \{c_{ij} \mid i \leq \frac{m+1}{2} \quad , \quad j \geq \frac{m+1}{2}\}$$

$$O_2^* = \{c_{ij} \mid i \geq \frac{m+1}{2} \quad , \quad j \leq \frac{m+1}{2}\} \quad .$$

Let $O^* = O_1^* \cup O_2^*$ then,

$$O_1^* \cap O_2^* = \begin{cases} \emptyset \text{ if } m \text{ is even} \\ c_{m+1/2\ ,\ m+1/2} \text{ if } m \text{ is odd} \end{cases} \quad .$$

These two sets of entries are symmetric with respect to the principal diagonal

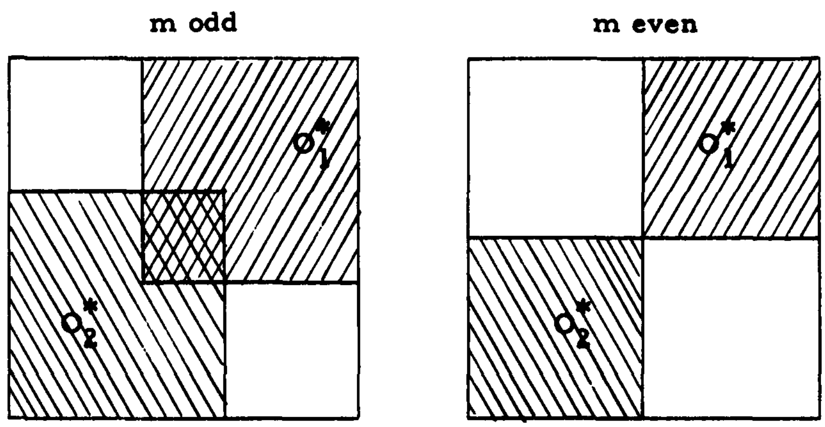

Figure 1 – Optimal Region.

Let $O_1 = \{x_{ij} | x_{ij} \in O_1 \text{ iff } c_{ij} \in O_1^*\}$

$O_2 = \{x_{ij} | x_{ij} \in O_2 \text{ iff } c_{ij} \in O_2^*\}$.

Then, $O = O_1 \cup O_2$ and it is called the <u>optimal region</u>.

NOTE: If we were minimizing $\Sigma c_{ij}x_{ij}$ then the optimal solution would be unique and would be $x = (x_{11}, x_{22}, x_{33}, \ldots, x_{mm})$ which is easy to see because all the elements of the principal diagonal are zero and the other elements are strictly positive.

IV. Properties of the Reordering Matrix

1. For all $c_{rs} \notin O^*$

   either all $c_{rk} \in O^*$ are greater than or equal to c_{rs} or all $c_{ks} \in O^*$ are greater than or equal to c_{rs}.

2. For any submatrix of C of the form

$$C_s = \begin{bmatrix} c_{ij} & c_{i,j+1} \\ c_{i+1,j} & c_{i+1,j+1} \end{bmatrix}$$

   such that $i \neq j$

   then $c_{ij} + c_{i+1,j+1} = c_{i+1,j} + c_{i,j+1}$. In fact this property is true for any submatrix chosen so that all its elements

$$C_s = \begin{bmatrix} c_{ij} & c_{i,j+s} \\ c_{i+r,j} & c_{i+r,j+s} \end{bmatrix}$$

   are all below the principal diagonal or above the principal diagonal.

3. For any submatrix of C of the form

$$C_r = \begin{bmatrix} c_{ij} & c_{i,j+s} \\ c_{i+r,j} & c_{i+r,j+s} \end{bmatrix}$$

   such that $c_{i+r,j}$ is an entry below the principal diagonal of C and $c_{i,j+s}$ is an entry above the principal diagonal. Then

$$c_{ij} + c_{i+r,j+s} < c_{i+r,j} + c_{i,j+s}$$

PROOF: When $c_{i+r,j}$ is below the principal diagonal of C, it implies that

$$i + r > j \Rightarrow i - j + r > 0 \Rightarrow r > j - i \quad . \tag{3}$$

When $c_{i,j+s}$ is above the principal diagonal of C, it implies that

$$j + s > i \Rightarrow i - j - s < 0 \Rightarrow s > i - j \quad . \tag{4}$$

$c_{ij} + c_{i+r,j+s} < c_{i+r,j} + c_{i,j+s}$ can be written as follows:

$$|i - j| + |i - j + r - s| < |i - j + r| + |i - j - s| \quad .$$

From (3) and (4), we get

$$\begin{aligned} |i - j| + |i - j + r - s| &< (i - j + r) - (i - j - s) \quad , \\ |i - j| + |i - j + r - s| &< r + s \quad . \end{aligned} \tag{5}$$

Here we can distinguish various cases. We will prove it for one case. Assume $i > j$ and $r > s$; (5) then becomes

$$i - j + i - j + r - s < r + s$$

then

$$2i - 2j < 2s$$

or

$$i - j < s$$

which is relation (4).

It is possible to prove the optimality of any feasible solution in region O by using the theory of linear programming, or more specifically the assignment problem algorithm.[2] Let the simplex multipliers be

$$\pi = -i \quad \text{for the constraints} \quad \sum_j x_{ij} = 1 \qquad \text{for } i < \frac{m+1}{2}$$

$$= +i \quad \text{for the constraints} \quad \sum_j x_{ij} = 1 \qquad \text{for } i \geq \frac{m+1}{2}$$

$$\pi = -j \quad \text{for the constraints} \quad \sum_i x_{ij} = 1 \qquad \text{for } j < \frac{m+1}{2}$$

$$= +j \quad \text{for the constraints} \quad \sum_i x_{ij} = 1 \qquad \text{for } j \geq \frac{m+1}{2} .$$

If we price out, it is possible to see that the $\bar{c}_{ij} \in O^*$ are equal to zero and the other $\bar{c}_{ij} \notin O^*$ are ~~positive~~ negative. We are going to give an alternative proof which will use the properties of the reordering matrix and the fact that the solution has to be feasible.

V. Optimality Theorem

THEOREM: A feasible solution x^o is optimal if and only if all its components lie in O .

PROOF: Suppose $\hat{x}$ is afeasible solution but has at least one component which does not lie in O , we will then show that we can improve S .

Let us assume that $\hat{x}_{ij}$ is such that $i < \frac{m+1}{2}$, $j < \frac{m+1}{2}$ and $j \geq i$, i.e., in Figure 2, $\hat{x}_{ij}$ lies in U . In order to be feasible x has at least a component, say $\hat{x}_{k\ell} \in V$, because it is impossible to "cover"

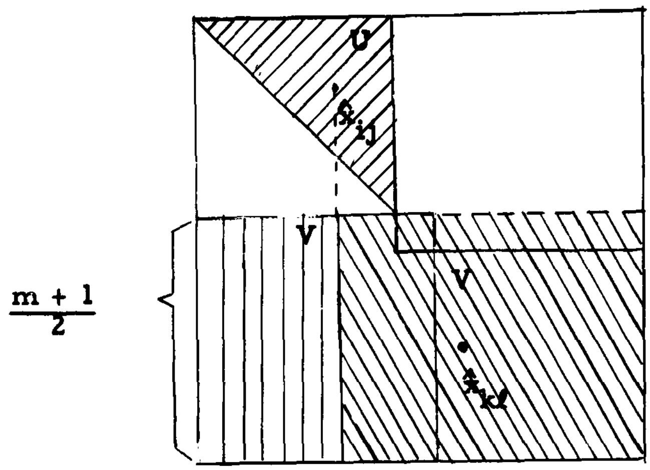

Figure 2

the $\left[\frac{m+1}{2}\right]$ * last rows with "selections" only done in $\mathbf{V}$, as $\mathbf{V}$ has less than $\left[\frac{m+1}{2}\right]$ columns (by assumption on $\hat{x}_{ij}$). We can then find a new feasible solution by replacing $\hat{x}_{ij}$ and $\hat{x}_{kl}$ by x_{il} and x_{kj} and by property 3 , S will be improved.

We have proved that for any feasible solution which has one or more components outside O , it can be improved. So, we can produce an iterative procedure which will increase S , as long as x has a component which does not lie in O .

If $x^o \in O$, it is not possible to improve the value of S because the only acceptable substitutions are of the form

$$x^o_{ij} \quad \text{and} \quad x^o_{i+r,\,j+s}$$

by

$$x^o_{i+r,\,j} \text{ and } x^o_{i,\,j+s} \qquad (\in O)$$

*$[\alpha]$ = greatest integer contained in α .

or repeated substitutions of that form. But by property 2 we know that the value of S will not change. If one of the components of our substitutions does not lie in O then we have proved that we can improve the value of S.

COROLLARY: All the feasible solutions in region O are optimal.

PROOF: By property 2 of the reordering matrix, from an optimal solution x^o, we can find new optimal solutions $\bar{x}$ by repeated substitutions of the form:

$$x^o_{ij} = x^o_{i+r,j+s} = 1 \quad , \quad x^o_{i+r,j} = x^o_{i,j+s} = 0 \tag{6}$$

by

$$\bar{x}_{ij} = \bar{x}_{i+r,j+s} = 0 \quad , \quad \bar{x}_{i+r,j} = x_{i,j+s} = 1 \quad ,$$

as long as $x_{i+r,j}$ and $x_{i,j+s}$ belong to O.

NOTE: By repeated substitution of the form (6) it is possible to reproduce all m-tuples $a \in \mathscr{X}$ in matrix C.

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